Time-Optimal Control of State Constrained Linear Systems

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Recently Schlossmacher (1973) developed an algorithm for the calculation of deadbeat control policies for a special class of continuous linear systems. The purpose of this note is to extend the fundamental idea of Schlossmacher to make it applicable for a more general type of state constraint. Specifically, we are interested in determining the control policy which will accomplish the transition from one state of operation to another in minimum time and so that some state variables are constrained not to exceed some levels.

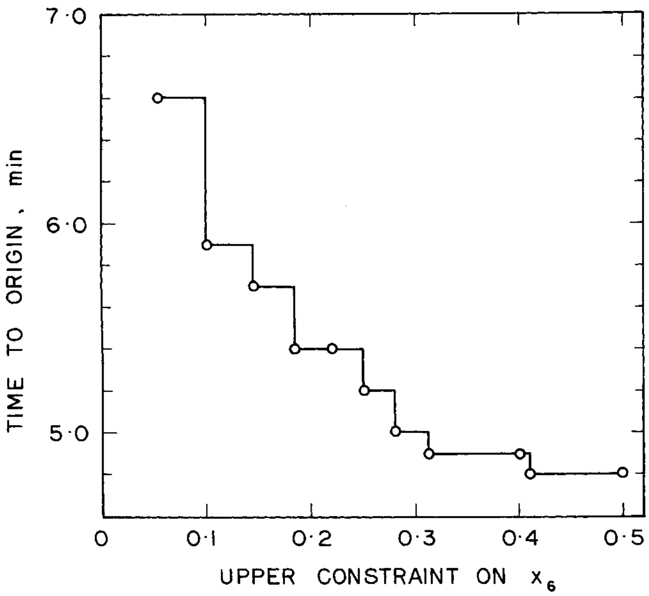

Fig. 1. Effect of upper constraint on x_6 on time to reach the origin.

For example, if we consider the 6-plate gas absorber which was used for illustration by Schlossmacher (1973) and described in detail by Lapidus and Luus (1967), we may wish to place the constraint:

$$\left.\begin{array}{l} x_1 \leq K_1 \\ x_6 \leq K_6 \end{array}\right\} \quad (1)$$

on the concentrations of solute on the first and the sixth plate while the system is transferred from a given state to the origin. Schlossmacher considered the case where the upper constraints K_1 and K_6 were taken to be 0.0. We wish to present an effective way to solve the minimum-time problem for any positive values of K_1 and K_6.

The state equation defining the gas absorber as given by Lapidus and Luus (1967) is

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \quad (2)$$

where **A** is a (6×6) tridiagonal matrix of elements $a = 0.538998$, $b = -1.173113$ and $c = 0.634115$ and **B** is a (6×2) matrix with $b_{11} = a$ and $b_{62} = c$. The control **u** is constrained by $0 \leq u_1 \leq 1.0$, $-0.4167 \leq u_2 \leq 0.972$.

It is clear that to keep x_1 from changing, it is necessary to choose

$$u_1 = -\,(bx_1 + cx_2)/a \quad (3)$$

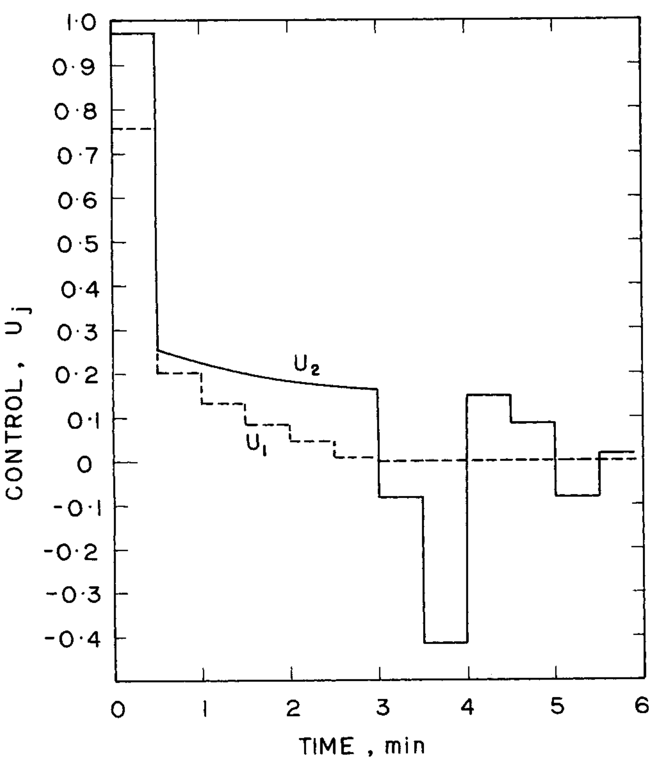

Fig. 2. Time-optimal control policy for the gas absorber with $K_6 = 0.10$ and $T = 0.5$. Integration step-size = 0.1.

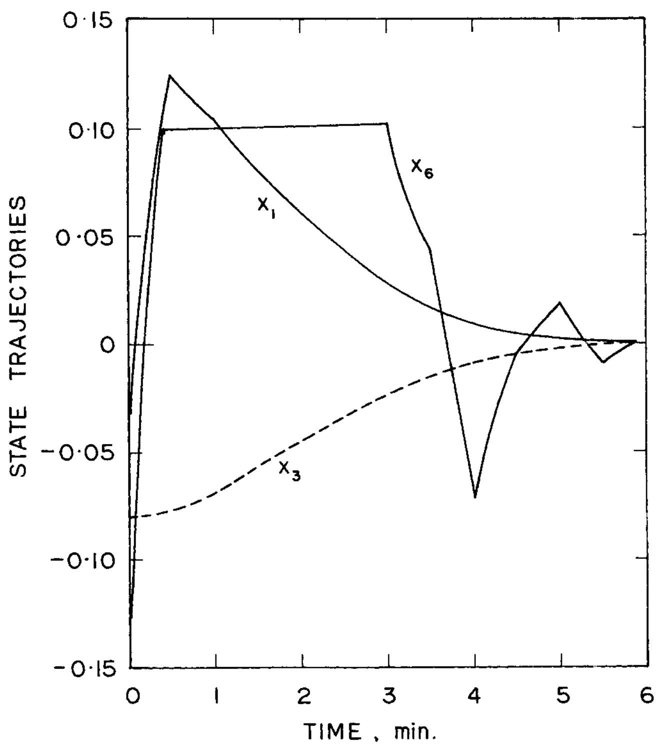

Fig. 3. State trajectories resulting from application of control policy in Figure 2.

and to keep x_6 from changing, it is necessary to choose

$$u_2 = -(ax_5 + bx_6)/c \qquad (4)$$

These are the two equations proposed by Schlossmacher (1973) to keep x_1 and x_6 at 0.000.

Here, it is proposed to use Equation (3) if x_1 is at its upper bound and if the control law calculated from the unconstrained optimal control policy would tend to cause the upper bound on x_1 to be surpassed. Similarly, it is proposed to use Equation (4) if x_6 is at its upper bound and if the control calculated from the unconstrained optimal control policy would tend to cause the upper bound on x_6 to be violated.

In order to obtain a feedback control law for the time-optimal control problem, the procedure presented by Bennett and Luus (1971) may be used. Here, however, a different optimization procedure is used to obtain the best value of the weighting matrix in the quadratic form. The optimization procedure of Luus and Jaakola (1973) is used instead of the procedure developed by Rosenbrock (1960) since it gives considerably better results.

The time-optimal control problem was solved with $K_1 = 0.125$ and various values for K_2. As is shown in Figure 1, it takes longer to reach the origin when K_2 is reduced, that is, when the upper constraint on x_6 is stiffened.

In Figure 2 is shown the control policy when K_6 is taken to be 0.10 and Figure 3 shows the trajectories of x_1, x_3, and x_6 corresponding to this control action. It can be readily seen that the origin is reached rapidly and x_6 is kept below its allowed constraint at all times. Therefore, if control action has a direct effect on the time derivative of a state variable, that state variable may be constrained and the time-optimal problem can be readily handled by the proposed procedure.

It should also be noted that in the absence of constraints the present procedure allows the origin to be reached in 4.8 min. (with a sampling time of 0.5 min.). This is considerably better than 6.0 min. reported by Bashein (1971) for the same problem. Therefore, the procedure which is expected to give only suboptimal results has yielded better results than are possible with a modified linear programming procedure.

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Measurement of Yield Stresses in Thermoplastic Polymer Melts by the Capillary Rise Method

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Wolpert and Wojtkowiak (1972) have recently reported that the leveling of a polymer powder paint is possibly inhibited by a very low level yield stress property of the melt. The data reported in the Wolpert paper can be explained by a yield stress of about 30 dynes/cm^2 if the melt surface tension is near 30 dynes/cm. Since these very low stress levels for melts are beyond the strain rate capabilities of ordinary viscometric and rheogoniometric devices, we decided to try a more sensitive tool—the capillary rise experiment. We report analyses of the equilibrium rise of a yield-stress fluid and the rise rate of a Bingham fluid as well as the results of one experiment.

THEORY

The equilibrium rise of a Newtonian fluid is characterized by the balance of surface tension forces against gravitational forces. This balance reads, for the configuration in Figure 1:

$$2\sigma\pi R \cos\beta = \rho g \pi R^2 l_2 \sin\theta \qquad (1)$$

where σ is the fluid surface tension, β is the contact angle between the capillary surface and the fluid, ρ is the fluid density, and g is the acceleration of gravity. Equation (1) may be solved for the capillary rise l_2

$$l_2 = \frac{2\sigma \cos\beta}{\rho g R \sin\theta} \qquad (2)$$

Thus, for a given fluid l_2 may be made as large as desired by decreasing the capillary diameter and decreasing the angle θ.

If the fluid possesses a yield stress, the equilibrium capillary rise will be limited by an additional force. Surface tension forces will be balanced against the sum of gravitational and yield forces, and in place of Equation

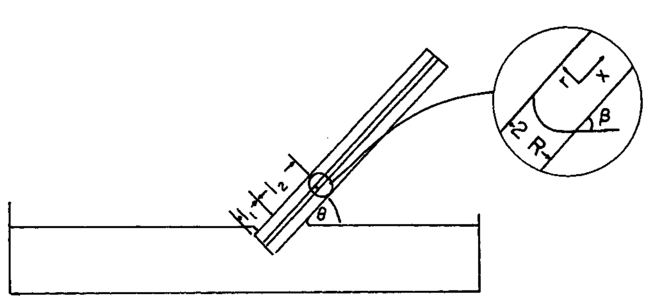

Fig. 1. Schematic of capillary rise